The maximum number of q-colorings in graphs

Jie Ma*

Abstract

We study an old problem of Linial and Wilf to find the graphs with n vertices and m edges which maximize the number of proper q-colorings of their vertices. In a breakthrough paper, Loh, Pikhurko and Sudakov reduced the problem to an optimization problem and solved asymptotically for many ranges of parameters. We show that for any instance of parameters, the optimization problem always has a solution which corresponds to either a complete multipartite graph or a graph obtained from complete multipartite graph by removing edges of two bipartite subgraphs. There are examples showing that each of the two classes of graphs is optimal for certain instance and hence both of them are not redundant.

We then apply this structural result of optimal graphs to general instances, including a conjecture of Lazebnik from 1989 which asserts that for any $q \geq s \geq 2$ the Turán graph $T_s(n)$ has the maximum number of q-colorings among all graphs with the same number of vertices and edges. We disprove this conjecture by providing infinity many counterexamples (s, q) for $s \geq 11$ and $s + 8 \leq q \leq 2s - 3$. On the positive side, we show that when $q \geq \Omega(s^2)$, the Turán graph $T_s(n)$ indeed achieves the maximum number of q-colorings.

Joint work with Humberto Naves and Benny Sudakov.

*Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: jiema@math.ucla.edu.