Logical limit laws for random graphs from minor closed classes

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Let $\mathcal{G}$ be a minor closed class of graphs, and let $\mathcal{G}_n$ denote the set of (labelled) graphs from $\mathcal{G}$ on exactly n vertices, and let $\mathcal{C}_n$ denote set of (labelled) connected graphs from $\mathcal{G}$. Let G_n be a graph chosen uniformly at random from $\mathcal{G}_n$ and let C_n be chosen uniformly at random from $\mathcal{C}_n$. We say that $\mathcal{G}$ is *addable* if 1) whenever $G \in \mathcal{G}$ adding an abritrary edge between distinct components produces a graph that is still $\in \mathcal{G}$, and 2) $G \in \mathcal{G}$ iff. every component of G is. Examples include the class of all forests, and the class of all planar graphs. Non-examples include graphs embeddable on some surface other than the plane/sphere, and forests of caterpillars.

A *first order (FO) property* is a graph property that can be expressed by a logic sentence using the quantifiers $\forall, \exists$ with variables ranging over the vertices of the graph, the logical connectives $\wedge, \vee, \neg$, etc., brackets and the relation symbols $=, \sim$, where $x \sim y$ means x and y are connected by an edge. Triangle-freeness is an example of a FO property as it can be written as $\neg \exists x, y, z : (x \sim y) \wedge (x \sim z) \wedge (y \sim z)$. A *monadic second order (MSO) property* is defined similarly, but now we are also allowed to quantify over subsets of the vertices, and we can ask about membership of these sets. Connectedness is an example of a MSO property as it can be written as $\forall X : \neg(\exists x : x \in X) \vee (\forall x : x \in X) \vee (\exists x, y : (x \in X) \wedge \neg(y \in X) \wedge (x \sim y))$ (for every partition into two non-empty parts, there is an edge going across).

We show that, if $\mathcal{G}$ is an addable minor closed class, then

$$\lim_{n \to \infty} \mathbb{P}(C_n \text{ satisfies } \varphi) \in \{0, 1\}, \tag{1}$$

for every MSO φ. This provides an analogue of a classical result by Glebskii et al.'69 and independently Fagin'76 on FO properties of the Erdős-Rényi random graph.

Note that (1) is for the *connected* random graph from $\mathcal{G}$. It is in fact easy to prove that it will fail if we replace C_n by G_n. We are however able to show that

$$\lim_{n \to \infty} \mathbb{P}(G_n \text{ satisfies } \varphi) \text{ exists},$$

for every MSO φ.

The same conclusions hold if $\mathcal{G}$ is the class of all graphs embeddable on a given surface S provided we weaken MSO to FO, and in fact the limiting probabilities do not depend on the choice of the surface S.

In the cases of forests and planar graphs, we are also able give an explicit description of the closure of the set of all limiting probabilities (this happens to be a union of 4 resp. 108 disjoint intervals) and we can give examples of non-addable graph classes that exhibit a very different behaviour.

(Based on joint work with P. Heinig, M. Noy and A. Taraz)