When does zero-one k-law hold?

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We study asymptotical behaviour of the probabilities of first-order properties for Erdős–Rényi random graphs $G(N,p)$. It was proved by Y.V. Glebskii, D.I. Kogan, M.I. Liagonkii and V.A. Talanov [1] in 1969 (and independently in 1976 by R. Fagin [2]) that for any first order property L either "almost all" graphs satisfy this property as N tends to infinity or "almost all" graphs don't satisfy the property. In other words, if p doesn't depend on N, then for any first-order property L either the random graph satisfies the property L almost surely or it doesn't satisfy (in such cases the random graph is said to *obey zero-one law*). We consider the probabilities $p=p(N)$, where $p(N)=N^{-\alpha}$, $N\in\mathbb{N}$, for $\alpha\in(0,1)$. The zero-one law for such probabilities was proved by S. Shelah and J.H. Spencer [3]. When $\alpha\in(0,1)$ is rational the zero-one law in ordinary sense for these graphs doesn't hold.

Let k be a positive integer. Denote by $\mathcal{L}_k$ the class of the first-order properties of graphs defined by formulae with quantifier depth bounded by the number k (the sentences are of a finite length). Let us say that the random graph obeys *zero-one k-law*, if for any first-order property $L\in\mathcal{L}_k$ either the random graph satisfies the property L almost surely or it doesn't satisfy. We consider set $S_k=\left[0,\frac{1}{k-2}\right]\cup\left(1-\frac{1}{2^{k-1}},1\right]$ and prove that random graph $G(N,N^{-\alpha})$ obeys zero-one k-law for any $\alpha\in\left[0,\frac{1}{k-2}\right)$ and for any $\alpha=1-\frac{1}{2^{k-1}+\beta}$, where $\beta\in(0,\infty)\setminus\mathcal{Q}$, $\mathcal{Q}$ is the set of positive rational numbers with numerator less than or equal to 2^{k-1}.

We find subset $\tilde{S}_k\subset S_k$ such that random graph $G(N,N^{-\alpha})$ does not obey zero-one k-law for any $\alpha\in\tilde{S}_k$. Note that numbers $\frac{1}{k-2},1-\frac{1}{2^{k-1}+1},1-\frac{1}{2^{k-1}+2},\ldots,1-\frac{1}{2^k-2}$ are in $\tilde{S}_k$.

References

[1] Y.V. Glebskii, D.I. Kogan, M.I. Liagonkii, V.A. Talanov, *Range and degree of realizability of formulas the restricted predicate calculus*, Cybernetics **5**: 142-154. (Russian original: Kibernetica **2**, 17-27).

[2] R. Fagin, *Probabilities in finite models*, J. Symbolic Logic **41**: 50–58, 1976.

[3] S. Shelah, J.H. Spencer, *Zero-one laws for sparse random graphs*, J. Amer. Math. Soc. **1**: 97–115, 1988.